第四节 定积分的换元法与分部积分法

一、定积分的换元法

二、定积分的分部积分法

一、定积分的换元法

定理1. 设函数 $f(x)\in C[a,b]$, 单值函数 $x=\varphi(t)$ 满足:

1) 函数 $\varphi(t)$ 在区间 $[\alpha,\beta]$ 上有连续的导数 $\varphi'(t)$;

2) $\varphi(\alpha)=a,\varphi(\beta)=b$, 在 $[\alpha,\beta]$ 上 $a\le\varphi(t)\le b$,

则 $$\int_a^b f(x)\mathrm{d}x=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t$$

证: 所证等式两边被积函数都连续, 因此积分都存在, 且它们的原函数也存在. 设 $F(x)$ 是 $f(x)$ 的一个原函数, 则 $F[\varphi(t)]$ 是 $f[\varphi(t)]\varphi'(t)$ 的原函数, 因此有

$$\int_a^b f(x)\mathrm{d}x=F(b)-F(a)=F[\varphi(\beta)]-F[\varphi(\alpha)]$$

$$\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t=\Big[F[\varphi(t)]\Big]_\alpha^\beta$$

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t$$

三定限　一代　二换

说明:

1) 当$\beta < \alpha$, 即区间换为$[\beta,\alpha]$时， 定理 1 仍成立 .

2) 必须注意换元必换限 , 原函数中的变量不必代回 .

3) 换元公式也可反过来使用 , 即

$$\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t = \int_\alpha^\beta f[\varphi(t)]\,\mathrm{d}\varphi(t) = \int_a^b f(x)\,\mathrm{d}x$$

配元不换限　令 $x = \varphi(t)$

例1. 计算$\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x \quad (a > 0).$ 三角代换

解: 令$x = a\sin t$, 则$\mathrm{d}x = a\cos t\,\mathrm{d}t$, 且

当$x = 0$时, $t = 0$; $x = a$时, $t = \frac{\pi}{2}$

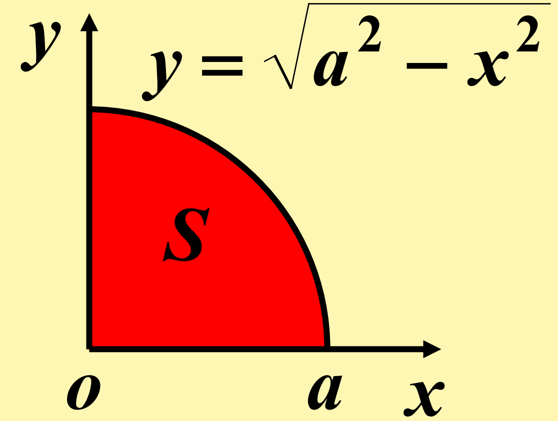
$\therefore$ 原式 $= a^2\int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)\,\mathrm{d}t$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}$$

用几何意义?



例2. 计算$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x$.

根式代换

解: 令$t=\sqrt{2x+1}$,则$x=\frac{t^2-1}{2}$, $\mathrm{d}x=t\,\mathrm{d}t$, 且

当$x=0$时,$t=1$; $x=4$时, $t=3$.

$$\therefore \quad 原式=\int_1^3 \frac{\frac{t^2-1}{2}+2}{t}\,t\,\mathrm{d}t=\frac{1}{2}\int_1^3 (t^2+3)\,\mathrm{d}t$$

$$=\frac{1}{2}\left(\frac{1}{3}t^3+3t\right)\Big|_1^3=\frac{22}{3}$$

例3. 计算 $\int_{-3}^{-2} \frac{1}{x^2\sqrt{x^2-1}} \mathrm{d}x$.

倒代换

解: 令 $x = \frac{1}{t}$, 则 $\mathrm{d}x = -\frac{1}{t^2}\mathrm{d}t$, 且

当 $x=-3$ 时, $t=-\frac{1}{3}$; $x=-2$ 时, $t=-\frac{1}{2}$.

$$原式 = \int_{-\frac{1}{3}}^{-\frac{1}{2}} \frac{t}{\sqrt{1-t^2}} \mathrm{d}t = -\frac{1}{2}\int_{-\frac{1}{3}}^{-\frac{1}{2}} (1-t^2)^{-\frac{1}{2}} \mathrm{d}(1-t^2)$$

$$= -\frac{1}{2} \cdot 2\sqrt{1-t^2}\,\Big|_{-\frac{1}{3}}^{-\frac{1}{2}} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2}$$

比较: $\int_{-2}^{-\sqrt{2}} \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$

例4. 设 $f(x)=\begin{cases} xe^{-x^2}, & x\ge 0 \\ \dfrac{1}{1+\cos x}, & -1\le x<0 \end{cases}$ 求 $\int_1^4 f(x-2)\,\mathrm{d}x$.

解：令 $t=x-2$，则 $\mathrm{d}x=\mathrm{d}t$，且

当 $x=1$ 时，$t=-1$； $x=4$ 时，$t=2$.

$$原式=\int_{-1}^{2} f(t)\,\mathrm{d}t=\int_{-1}^{2} f(x)\,\mathrm{d}x=\int_{-1}^{0}\frac{1}{1+\cos x}\,\mathrm{d}x+$$

$$\int_0^2 xe^{-x^2}\,\mathrm{d}x=\int_{-1}^{0}\frac{1}{2\cos^2\frac{x}{2}}\,\mathrm{d}x-\frac{1}{2}\int_0^2 e^{-x^2}\,\mathrm{d}(-x^2)$$

$$=[\tan\frac{x}{2}]_{-1}^{0}-[\frac{1}{2}e^{-x^2}]_0^2=\tan\frac{1}{2}-\frac{1}{2}e^{-4}+\frac{1}{2}$$

例5. 设$f(x)$是连续的以$T(>0)$为周期的周期函数，证明

(1) 对任何实数a，有 $\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$

(2) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

(3) $F(x) = \int_0^x f(t)\mathrm{d}t$的周期为$T \Leftrightarrow \int_0^T f(t)\mathrm{d}t = 0$

证明: (1) $\int_a^{a+T} f(x)\mathrm{d}x = \int_a^T f(x)\mathrm{d}x + \int_T^{a+T} f(x)\mathrm{d}x$

而 $\int_T^{a+T} f(x)\mathrm{d}x \xlongequal{x=T+u} \int_0^a f(u+T)\mathrm{d}u$

$= \int_0^a f(u)\mathrm{d}u = \int_0^a f(x)\mathrm{d}x$

$\therefore \int_a^{a+T} f(x)\mathrm{d}x = \int_a^T f(x)\mathrm{d}x + \int_0^a f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$

例5. 设$f(x)$是连续的以$T(>0)$为周期的周期函数，证明

(1) $\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$ (2) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

(3) $F(x) = \int_0^x f(t)\mathrm{d}t$的周期为$T \Leftrightarrow \int_0^T f(t)\mathrm{d}t = 0$

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(2) $\int_0^{nT} f(x)dx = \int_0^T f(x)dx + \int_T^{2T} f(x)dx + \cdots + \int_{(n-1)T}^{nT} f(x)dx$

$$= n\int_0^T f(x)dx$$

(3) $F(x+T) = \int_0^{x+T} f(t)\mathrm{d}t = \int_0^x f(t)\mathrm{d}t + \int_x^{x+T} f(t)\mathrm{d}t$

$$= F(x) + \int_0^T f(t)\mathrm{d}t$$

$$\int_0^T f(t)\mathrm{d}t = 0 \Longleftrightarrow F(x+T) \equiv F(x)$$

例6. 证明 $f(x)=\int_{x}^{x+\frac{\pi}{2}}|\sin x|\,\mathrm{d}x$是以$\pi$为周期的函数.

证明: $f(x)=\int_{x}^{x+\frac{\pi}{2}}|\sin u|\,\mathrm{d}u$

$$f(x+\pi)=\int_{x+\pi}^{x+\pi+\frac{\pi}{2}}|\sin u|\,\mathrm{d}u$$

令 $t=u-\pi$

$$=\int_{x}^{x+\frac{\pi}{2}}|\sin(t+\pi)|\,\mathrm{d}t$$

$$=\int_{x}^{x+\frac{\pi}{2}}|\sin t|\,\mathrm{d}t$$

$$=f(x)$$

$\therefore$ $f(x)$是以π为周期的周期函数.

例7. 设 $f(x)\in C[-a,a]$,

奇零偶倍

(1) 若 $f(-x)=f(x)$, 则 $\int_{-a}^{a}f(x)\mathrm{d}x=2\int_{0}^{a}f(x)\mathrm{d}x$

(2) 若 $f(-x)=-f(x)$, 则 $\int_{-a}^{a}f(x)\mathrm{d}x=0$

证: $\int_{-a}^{a}f(x)\mathrm{d}x=\int_{-a}^{0}f(x)\mathrm{d}x+\int_{0}^{a}f(x)\mathrm{d}x$

令 $x=-t$

$$=\int_{0}^{a}f(-t)\mathrm{d}t+\int_{0}^{a}f(x)\mathrm{d}x$$

$$=\int_{0}^{a}[f(-x)+f(x)]\mathrm{d}x$$

$$=\begin{cases}2\int_{0}^{a}f(x)\mathrm{d}x, & f(-x)=f(x)\text{时}\\ 0, & f(-x)=-f(x)\text{时}\end{cases}$$

例8. 计算 $\int_{-\pi}^{\pi}(\sqrt{1+\cos 2x}+|x|\sin x)\mathrm{d}x$

解: 原式 $=\int_{-\pi}^{\pi}\sqrt{1+\cos 2x}\mathrm{d}x+\int_{-\pi}^{\pi}|x|\sin x\mathrm{d}x$

偶函数　　奇函数

$$=2\int_{0}^{\pi}\sqrt{1+\cos 2x}\mathrm{d}x=2\int_{0}^{\pi}\sqrt{2}|\cos x|\mathrm{d}x$$

$$=2\sqrt{2}\left(\int_{0}^{\frac{\pi}{2}}\cos x\mathrm{d}x-\int_{\frac{\pi}{2}}^{\pi}\cos x\mathrm{d}x\right)$$

$$=2\sqrt{2}\left(\sin x\Big|_{0}^{\frac{\pi}{2}}-\sin x\Big|_{\frac{\pi}{2}}^{\pi}\right)$$

$$=4\sqrt{2}\quad =4\int_{0}^{\frac{\pi}{2}}\sqrt{2}\cos x\mathrm{d}x$$

结论: $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a} [f(-x)+f(x)]\mathrm{d}x$

$$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{a} f(-x)\mathrm{d}x = \frac{1}{2}\int_{-a}^{a} [f(-x)+f(x)]\mathrm{d}x$$

例9. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1-\sin x}\mathrm{d}x = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{1-\sin(-x)}+\frac{1}{1-\sin x}\right)\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2}{\cos^2 x}\mathrm{d}x = 2\int_{0}^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\mathrm{d}x = 2\tan x\Big|_{0}^{\frac{\pi}{4}} = 2$$

法二:

$$原式=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\mathrm{d}x$$

例8. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1-\sin x}\mathrm{d}x$

法三： 万能代换

令 $t=\tan\frac{x}{2}$，则 $\sin x=\frac{2t}{1+t^2}$, $\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$

$$\text{原式}=\int_{-\tan\frac{\pi}{8}}^{\tan\frac{\pi}{8}} \frac{1}{1-\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t$$

$\tan\frac{\pi}{8}=?$

$$\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}=\frac{\sin x}{1+\cos x}$$

例8. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1-\sin x}\mathrm{d}x$

法四：原式$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\cos(x+\frac{\pi}{2})}\mathrm{d}x = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2\cos^2(\frac{x}{2}+\frac{\pi}{4})}\mathrm{d}x$

法五：

$$1-2\sin\frac{x}{2}\cos\frac{x}{2}=\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2=\left[\sqrt{2}\sin(\frac{x}{2}-\frac{\pi}{4})\right]^2$$

原式$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2\sin^2(\frac{x}{2}-\frac{\pi}{4})}\mathrm{d}x=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \csc^2(\frac{x}{2}-\frac{\pi}{4})\mathrm{d}(\frac{x}{2}-\frac{\pi}{4})$

$$=-\cot(\frac{x}{2}-\frac{\pi}{4})\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\cot\frac{\pi}{8}-\cot\frac{3\pi}{8}=(\sqrt{2}+1)-(\sqrt{2}-1)$$

例10. 设 $f(x)$是连续函数,证明:

(1) $\int_0^{\frac{\pi}{2}} f(\sin x,\cos x)\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x,\sin x)\mathrm{d}x$

(2) $\int_0^{\pi} xf(\sin x)\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x$

证明:(1) 令$x=\frac{\pi}{2}-t \quad \Rightarrow \mathrm{d}x=-\mathrm{d}t,$

$$x=0\Rightarrow t=\frac{\pi}{2},\quad x=\frac{\pi}{2}\Rightarrow t=0,$$

$$\int_0^{\frac{\pi}{2}} f(\sin x,\cos x)\mathrm{d}x=-\int_{\frac{\pi}{2}}^{0} f[\sin(\frac{\pi}{2}-t),\cos(\frac{\pi}{2}-t)]\mathrm{d}t$$

$$=\int_0^{\frac{\pi}{2}} f[\cos t,\sin t]\mathrm{d}t=\int_0^{\frac{\pi}{2}} f(\cos x,\sin x)\mathrm{d}x$$

★ $\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x=\int_0^{\frac{\pi}{2}}\cos^n x\mathrm{d}x$

(2) $\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$

证明：(2) 令$x = \pi - t \quad \Rightarrow \mathrm{d}x = -\mathrm{d}\,t,$

$$x = 0 \Rightarrow t = \pi, \quad x = \pi \Rightarrow t = 0,$$

$$\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = -\int_{\pi}^{0}(\pi - t)f[\sin(\pi - t)]\,\mathrm{d}t$$

$$= \int_0^{\pi}(\pi - t)f(\sin t)\,\mathrm{d}t$$

$$= \pi\int_0^{\pi} f(\sin t)\,\mathrm{d}t - \int_0^{\pi} tf(\sin t)\,\mathrm{d}t$$

用此结论可计算 $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x = \frac{\pi^2}{4}$

例11. 计算 $I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}\mathrm{d}x$

解：利用 $\int_0^{\frac{\pi}{2}}f(\sin x,\cos x)\mathrm{d}x=\int_0^{\frac{\pi}{2}}f(\cos x,\sin x)\mathrm{d}x$

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x+\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}x=\frac{\pi}{2}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x=\frac{\pi}{4}$$

例12. 计算 $\int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt$ **P219-Ex 3**

解：原式 $= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos t+\sin t}{\cos t}\right)dt$

$$= \int_0^{\frac{\pi}{4}} \ln(\cos t+\sin t)dt - \int_0^{\frac{\pi}{4}} \ln(\cos t)dt$$

$$= \int_0^{\frac{\pi}{4}} \ln\sqrt{2}\left(\sin(t+\frac{\pi}{4})\right)dt - \int_0^{\frac{\pi}{4}} \ln(\cos t)dt$$

$$= \int_0^{\frac{\pi}{4}} \ln\sqrt{2}dt + \int_0^{\frac{\pi}{4}} \ln\left(\sin(t+\frac{\pi}{4})\right)dt - \int_0^{\frac{\pi}{4}} \ln(\cos t)dt$$

令 $\frac{\pi}{4}+t=\frac{\pi}{2}-u$ 即 $u=\frac{\pi}{4}-t$

$$= \frac{\pi}{8}\ln 2 - \int_{\frac{\pi}{4}}^{0} \ln(\cos u)du - \int_0^{\frac{\pi}{4}} \ln(\cos t)dt \quad = \frac{\pi}{8}\ln 2$$

二、定积分的分部积分法

定理2. 设 $u(x), v(x) \in C[a, b]$，则

$$\int_a^b \left(u(x)v'(x) + u'(x)v(x)\right)\mathrm{d}x = u(x)v(x)\Big|_a^b$$

$$\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\mathrm{d}x$$

例12. 计算 $\int_0^{\frac{\pi}{4}} \frac{x\mathrm{d}x}{1+\cos 2x}$

解: 原式 $= \int_0^{\frac{\pi}{4}} \frac{x\mathrm{d}x}{2\cos^2 x} = \frac{1}{2}\int_0^{\frac{\pi}{4}} x\mathrm{d}(\tan x)$

$$= \frac{1}{2}\left[x\tan x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x\mathrm{d}x$$

$$= \frac{\pi}{8} + \frac{1}{2}\left[\ln\cos x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{\ln 2}{4}$$

例13. 计算$I=\int_0^1 x\left(\int_1^{x^2}\frac{\sin t}{t}\mathrm{d}t\right)\mathrm{d}x$

解: 设 $f(x)=\int_1^{x^2}\frac{\sin t}{t}\mathrm{d}t$, 则 $f(1)=0$

$$f'(x)=\frac{\sin x^2}{x^2}\cdot 2x$$

$$I=\int_0^1 x\,f(x)d\,x=\int_0^1 f(x)d\left(\frac{x^2}{2}\right)$$

$$=\left[\frac{x^2}{2}f(x)\right]_0^1-\int_0^1\frac{x^2}{2}\cdot\frac{\sin x^2}{x^2}\cdot 2x\mathrm{d}x$$

$$=0-\int_0^1 x\sin x^2\mathrm{d}x=\frac{1}{2}[\cos x^2]_0^1=\frac{1}{2}(\cos 1-1)$$

例14. 证明$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$

$$=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & n\text{ 为偶数}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}, & n\text{ 为奇数}\end{cases}$$

证明:

$$I_n=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\cdot\sin x\,\mathrm{d}x=-\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,\mathrm{d}\cos x$$

$$=\underbrace{[-\cos x\cdot\sin^{n-1}x]\Big|_0^{\frac{\pi}{2}}}_{0}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,\mathrm{d}x$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)\,\mathrm{d}x$$

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)\,\mathrm{d}x$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x$$

由此得递推公式 $I_n = \dfrac{n-1}{n} I_{n-2}$

于是 $I_{2m} = \dfrac{2m-1}{2m} \cdot \dfrac{2m-3}{2m-2} \cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot I_0$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

而 $I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \dfrac{\pi}{2}$, $I_1 = \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = 1$

故所证结论成立 .

例15. 设 $f''(x)$ 在 $[0,1]$ 上连续，且 $f(0)=1$，$f(2)=3$，$f'(2)=5$，求 $\int_0^1 xf''(2x)\mathrm{d}x$。

解： $\int_0^1 xf''(2x)\mathrm{d}x = \frac{1}{2}\int_0^1 x\,\mathrm{d}\,f'(2x)$

$$= \frac{1}{2}\left[xf'(2x)\right]_0^1 - \frac{1}{2}\int_0^1 f'(2x)\mathrm{d}x$$

$$= \frac{1}{2}f'(2) - \frac{1}{4}\left[f(2x)\right]_0^1$$

$$= \frac{5}{2} - \frac{1}{4}\left[f(2) - f(0)\right] = 2$$